Logarithmic Functions

Question: What is the inverse of an exponential function? How do you solve for a variable that is in an exponent?

Find the inverse of $f(x) = 2^x$.

1. Replace $f(x)$ with y. $\quad y = 2^x$
2. Interchange x and y. $\quad x = 2^y$
3. Solve for y. $\quad y =$ the exponent to which we raise 2 to get x.
4. Replace y with $f^{-1}(x)$ $\quad f^{-1}(x) =$ the exponent to which we raise 2 to get x.

We need a new symbol to replace the words: "The exponent to which we raise 2 to get x":

$\log_2 x$ means "the exponent to which we raise 2 to get x."

Pronounced "the logarithm, base 2, of x" or "log, base 2, of x"

★LOGARITHMS ARE EXPONENTS!★

Logarithm: **$\log_b a$** means the ***exponent*** to which we raise b to get a.

- b is called the ***base*** of the logarithm (the number being raised to the exponent).
- a is called the ***argument*** of the logarithm (the number you get when you raise the base to the exponent).

The ***logarithmic function of base a***, where $a > 0$ and $a \neq 1$ is denoted by $y = \log_a x$ and is defined by

$$y = \log_a x \text{ if and only if } x = a^y.$$

Example: Change each exponential expression to an equivalent expression involving a logarithm.

a) $5^x = 625$ b) $x^3 = 64$ c) $3^2 = x$

$\log_5 625 = x$ $\quad \log_x 64 = 3$ $\quad \log_3 x = 2$

Example: Change each logarithmic expression to an equivalent expression involving an exponent.

a) $\log_3 x = 5$ b) $\log_e 5 = x$ c) $\log_m 2 = n$

$3^5 = x$ $\quad e^x = 5$ $\quad m^n = 2$

Evaluating Logarithms: It is helpful to replace "log" with the word "power".

- Instead of "$\log_2 8$," think "$\text{power}_2\, 8$." Ask yourself, what power of 2 equals 8?
  - The answer would be 3 because $2^3 = 8$.

Example: Find the exact value of

a) $\log_3 9$ b) $\log_2 32$ c) $\log_6 1$ d) $\log_5 \frac{1}{125}$ e) $\log_7 \sqrt{7}$

a) $3^? = 9$ $\quad \boxed{2}$

b) $2^? = 32$ $\quad \boxed{5}$

c) $6^? = 1$ $\quad \boxed{0}$

d) $5^? = \frac{1}{125}$ $\quad \boxed{-3}$

e) $7^? = \sqrt{7}$ $\quad \boxed{1/2}$

Domain of a Logarithmic Function

The logarithmic function $y = \log_a x$ is the inverse of the exponential function $y = a^x$.

Domain of the logarithmic function = Range of the exponential function $= (0, \infty)$

Range of the logarithmic function = Domain of the exponential function $= (-\infty, \infty)$

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $(0, \infty)$ Range: all real numbers

★ **Caution!** You can't take the log of zero or of a negative because it is impossible to get zero or a negative by raising a positive base to an exponent. **The argument of a logarithmic function must be greater than zero.**

Example: Find the domain of each logarithmic function

a) $f(x) = \log_2(x+3)$

$x + 3 > 0$

$\boxed{\{x \mid x > -3\}}$

b) $g(x) = \log_5\left(\dfrac{1+x}{1-x}\right)$

$\dfrac{1+x}{1-x} > 0$

x-int: -1

vert. asymp: $x = 1$

	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
x	-2	0	2
$f(x)$	$-\frac{1}{3}$	1	-3

$\boxed{(-1,1) \text{ or } \{x \mid -1 < x < 1\}}$

c) $h(x) = \log_{\frac{1}{2}}|x|$

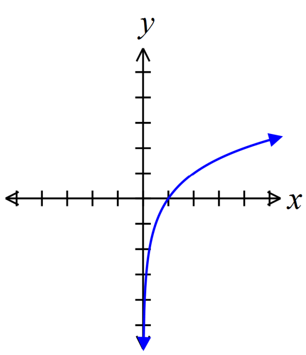
$|x| > 0$

The $|x|$ is positive unless $x = 0$

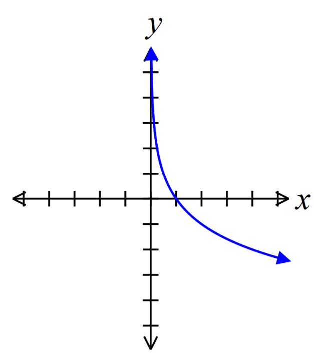
$\boxed{\{x \mid x \neq 0\}}$

Graphs of Logarithmic Functions

$f(x) = \log_a x, \ a > 1$



$f(x) = \log_a x, \ 0 < a < 1$



Properties of the Logarithmic Function $f(x) = \log_a x$

1. The domain is the set of all positive real numbers; the range is the set of all real numbers.
2. The x-intercept is 1. There is no y-intercept.
3. The y-axis $(x = 0)$ is a vertical asymptote of the graph.
4. The logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$. The function is one-to-one.
5. The graph of f contains the points $(1, 0)$, $(a, 1)$, and $\left(\dfrac{1}{a}, -1\right)$.
6. The graph of f is smooth and continuous, with no corners, gaps, or cusps.

★ **Note:** It is often easier to graph a logarithmic function if you rewrite it as an exponential function first.

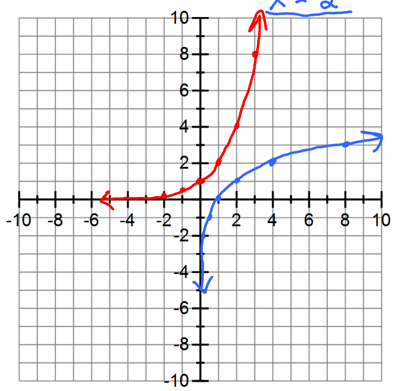
Graphing Logarithmic Functions:

1. Solve the equation for x by rewriting it as an exponential function.
2. Choose y-values, and plug them in to find the x-values.
3. Plot your points and connect them to form a smooth curve.

Examples:

a) Graph $y = 2^x$ and $y = \log_2 x$

$x = 2^y$



$y = 2^x$

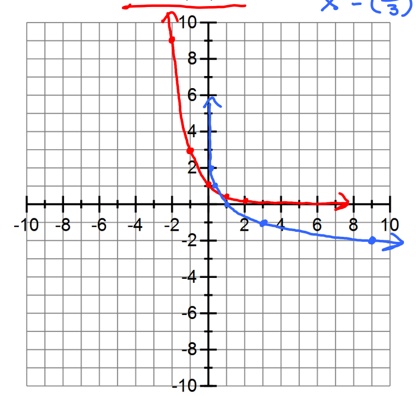
x	y
-2	1/4
-1	1/2
0	1
1	2
2	4

$x = 2^y$

x	y
1/4	-2
1/2	-1
1	0
2	1
4	2

b) Graph $y = \left(\frac{1}{3}\right)^x$ and $y = \log_{1/3} x$.

$x = (\frac{1}{3})^y$



$y = (\frac{1}{3})^x$

x	y
-2	9
-1	3
0	1
1	1/3
2	1/9

$x = (\frac{1}{3})^y$

x	y
9	-2
3	-1
1	0
1/3	1
1/9	2

Natural Logarithms: If the base of a logarithmic function is the number e, then we have the natural logarithm function (abbreviated ln). That is, $y = \ln x$ if and only if $x = e^y$.

Example: $f(x) = -\ln(x+3)$

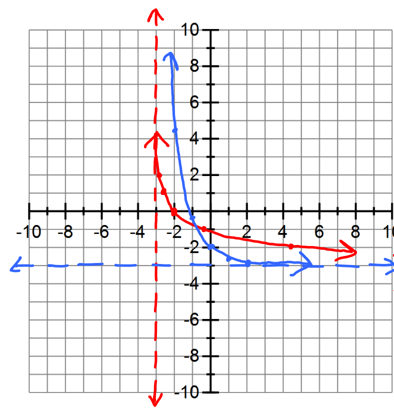
a) Find the domain of the logarithmic function.

b) Graph $f(x)$.

c) Find the range and vertical asymptote of f.

d) Find f^{-1}, the inverse of f.

e) Graph f^{-1}.



$y = -\ln(y+3)$

$x = e^{-y} - 3$

x	y
4.4	-2
-.3	-1
-2	0
-2.6	1
-2.9	2

$y = e^{-x} - 3$

x	y
-2	4.4
-1	-.3
0	-2
1	-2.6
2	-2.9

a) $x + 3 > 0$ $\boxed{\{x \mid x > -3\}}$

c) Range: $\boxed{\mathbb{R}}$

Vert. Asymp: $\boxed{x = -3}$

d) $x = -\ln(y+3)$

$-x = \ln(y+3)$

$e^{-x} = y + 3$

$y = e^{-x} - 3$

$\boxed{f^{-1}(x) = e^{-x} - 3}$

Common Logarithmic Function: If the base of a logarithmic function is the number 10, then we have the common logarithmic function. If the base a of the logarithmic function is not indicated, it is understood to be 10. That is, $y = \log x$ if and only if $x = 10^y$.

Example: $f(x) = 2\log(x-3)$

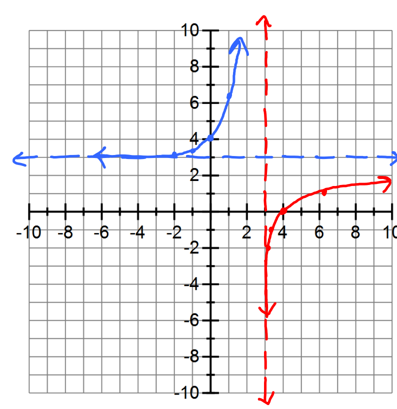
a) Find the domain of the logarithmic function.

b) Graph $f(x)$.

c) Find the range and vertical asymptote of f.

d) Find f^{-1}, the inverse of f.

e) Graph f^{-1}.



$y = 2\log(x-3)$

$x = 10^{y/2} + 3$

x	y
3.1	-2
3.3	-1
4	0
6.2	1
13	2

$y = 10^{x/2} + 3$

x	y
-2	3.1
-1	3.3
0	4
1	6.2
2	13

a) $x - 3 > 0$ $\boxed{\{x \mid x > 3\}}$

c) Range: $\boxed{\mathbb{R}}$

Vert. Asymp: $\boxed{x = 3}$

d) $x = 2\log(y-3)$

$\frac{x}{2} = \log(y-3)$

$10^{x/2} = y - 3$

$y = 10^{x/2} + 3$

$\boxed{f^{-1}(x) = 10^{x/2} + 3}$

Solving Logarithmic Equations
Many equations can be solved by rewriting logarithms as exponential functions or rewriting exponential functions as logarithms.

★ When solving logarithmic equations, remember that in the expression $\log_a M$, a and M must be positive and $a \neq 1$. Be sure to check each solution in the original equation and discard any that are extraneous.

Examples: Solve the logarithmic equations

a) $\log_3(3x-2)=2$

$3x-2>0$
$x>2/3$

$3x-2=3^2$
$3x-2=9$
$3x=11$
$\boxed{x=\frac{11}{3}}$

b) $\log_x\left(\frac{1}{8}\right)=3$

$x^3=\frac{1}{8}$
$\sqrt[3]{x^3}=\sqrt[3]{1/8}$
$\boxed{x=\frac{1}{2}}$

c) $10^{2x-7}=3$

$\log_{10}3=2x-7$
$2x=\log 3+7$
$\boxed{x=\frac{\log 3+7}{2}\approx 3.74}$

d) $e^{3x-2}=7$

$\log_e 7=3x-2$
$\ln 7+2=3x$
$\boxed{x=\frac{\ln 7+2}{3}\approx 1.32}$

e) $\log_2(x^2+2x)=3$

$x^2+2x>0$
$x(x+2)>0$
Domain: $(-\infty,-2)\cup(0,\infty)$

$2^3=x^2+2x$
$x^2+2x=8$
$x^2-2x-8=0$
$(x-4)(x+2)=0$
$\boxed{x=4}$ or $x\cancel{=-2}$

f) $4e^{x+1}=5$

$e^{x+1}=\frac{5}{4}$
$x+1=\ln\left(\frac{5}{4}\right)$
$\boxed{x=\ln\left(\frac{5}{4}\right)-1\approx -0.777}$

Example: The blood alcohol concentration (BAC) is the concentration of alcohol in a person's bloodstream. The relative risk of having an accident while driving a car is given by the equation $R=e^{kx}$, where R is the relative risk (how many times more likely a person with a certain BAC is to have a car accident than a person who has not been drinking), x is the BAC (expressed as a percentage), and k is a constant.

a) If the relative risk is 1.4 when the blood concentration is 0.02%, find k.

$1.4=e^{.02k}$
$\ln 1.4=.02k$
$k=\frac{\ln 1.4}{.02}\approx\boxed{16.8236}$

b) Using k from part a), find the relative risk if the blood alcohol concentration is 0.17%.

$R=e^{(16.8236)(0.17)}=\boxed{17.5}$

c) What BAC corresponds to a relative risk of 100?

$100=e^{16.8236x}$
$\ln 100=16.8236x$
$x=\frac{\ln 100}{16.8236}=\boxed{0.27\%}$